

GLOBAL WELL-POSEDNESS FOR 2D RADIAL SCHRÖDINGER MAPS INTO THE SPHERE.

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ABSTRACT. We prove global well-posedness for a cubic, non-local Schrödinger equation with radially-symmetric initial data in the critical space $L^2(\mathbb{R}^2)$, using the framework of Kenig-Merle and Killip-Tao-Visan. As a consequence, we obtain a global well-posedness result for Schrödinger maps from \mathbb{R}^2 into \mathbb{S}^2 (Landau-Lifshitz equation) with radially symmetric initial data (with no size restriction).

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1. INTRODUCTION AND MAIN RESULTS

The *Schrödinger map* equation

$$\vec{u}_t = \vec{u} \times \Delta \vec{u} \quad (1.1)$$

for maps $\vec{u}(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ into the 2-sphere

$$\vec{u}(\cdot, t) : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{S}^2 := \{ \vec{u} \in \mathbb{R}^3 \mid |\vec{u}|^2 = u_1^2 + u_2^2 + u_3^2 = 1 \}$$

arises as a continuum model of a ferromagnet, where it is known as the *Hesienberg model*, or (a special case of) the *Landau-Lifshitz equation* [12, 11]. From a geometric viewpoint, the Schrödinger map equation is a generalization of the (free) Schrödinger

equation with the (flat) target space \mathbb{C} replaced by a (curved) Kähler manifold, in this case \mathbb{S}^2 . To see this, it is helpful to re-write (1.1) as

$$\vec{u}_t = -J^{\vec{u}} \mathcal{E}'(\vec{u}) \quad (1.2)$$

where

$$\mathcal{E}(\vec{u}) = \frac{1}{2} \int_{\Omega} |\nabla \vec{u}|^2 dx$$

is the *energy* of the map $\vec{u}(\cdot, t)$,

$$\mathcal{E}'(\vec{u}) = -\Delta \vec{u} - |\nabla \vec{u}|^2 \vec{u}$$

is the gradient of \mathcal{E} (taking into account the geometric constraint $|\vec{u}| \equiv 1$), and

$$J^{\vec{u}} : \xi \mapsto \vec{u} \times \xi$$

is a complex structure ($\pi/2$ -rotation) on the tangent space

$$T_{\vec{u}}\mathbb{S}^2 = \{ \vec{\xi} \in \mathbb{R}^3 \mid \vec{u} \cdot \vec{\xi} = 0 \}$$

to the sphere at $\vec{u} \in \mathbb{S}^2$. Thus (1.1) (or (1.2)) is a natural analogue of the Schrödinger equation for maps into \mathbb{S}^2 , in the same way that the harmonic map heat-flow $\vec{u}_t = \Delta \vec{u} + |\nabla \vec{u}|^2 \vec{u}$ is an analogue of the heat equation, and the wave map equation $\vec{u}_{tt} = \Delta \vec{u} + (|\nabla \vec{u}|^2 - |\vec{u}_t|^2) \vec{u}$ is an analogue of the wave equation.

We take here $\Omega = \mathbb{R}^2$, and consider the Cauchy problem with initial data in a Sobolev space:

$$\begin{cases} \vec{u}_t = \vec{u} \times \Delta \vec{u} \\ \vec{u}(x, 0) = \vec{u}_0(x), \quad \vec{u}_0 - \widehat{k} \in H^k(\mathbb{R}^2) \end{cases} \quad (1.3)$$

(note that since $|\vec{u}| \equiv 1$, we must subtract a point on the sphere – here arbitrarily chosen to be $\widehat{k} = (0, 0, 1)$ – in order to have spatial decay). For smooth solutions, the conservation of energy

$$\mathcal{E}(\vec{u}(t)) = \frac{1}{2} \|\nabla \vec{u}(t)\|_{L^2(\mathbb{R}^2)}^2 \equiv \mathcal{E}(\vec{u}_0) \quad (1.4)$$

follows immediately from the Hamiltonian form (1.2), and the conservation law

$$\|\vec{u}(t) - \widehat{k}\|_{L^2(\mathbb{R}^2)}^2 \equiv \|\vec{u}_0 - \widehat{k}\|_{L^2(\mathbb{R}^2)}^2 \quad (1.5)$$

(which in Hamiltonian terms comes from invariance of the energy under rotations of \mathbb{S}^2 about the \widehat{k} axis) is easily checked. On \mathbb{R}^2 , this problem is *energy critical*, since the scaling $\vec{u}(x, t) \mapsto \vec{u}(\lambda x, \lambda^2 t)$, which preserves solutions of the Schrödinger map equation, also preserves the energy:

$$\mathcal{E}(\vec{u}(\lambda \cdot)) = \mathcal{E}(\vec{u}(\cdot)).$$

We will prove global well-posedness for (1.3) in the *radial* case, and for $k = 2$:

Theorem 1. *Assume $\vec{u}_0 = \vec{u}_0(r) \in \widehat{k} + H^2(\mathbb{R}^2)$, $r = |x|$. Then (1.3) has a unique global solution $\vec{u} \in L_{loc}^\infty([0, \infty); H^2(\mathbb{R}^2))$.*

To put this in context, the recent announcement [14] of the construction of (non-radial) finite-time blow-up solutions of (1.3) shows we should not expect global well-posedness for *all* (even smooth) data. More generally, experience with wave maps and harmonic map heat-flow suggests that a key to singularity formation is the presence of non-trivial static solutions – that is, harmonic maps – and a natural conjecture is that solutions with energy below that of any non-trivial harmonic map are global, a conjecture which has been proved for harmonic map heat-flow [20] and for wave maps [18, 19, 8], but not yet for Schrödinger maps. Theorem 1 is consistent with this general picture, since there are no non-trivial, *radial* harmonic maps into \mathbb{S}^2 . We mention a few more related results. *Equivariant* Schrödinger maps of topological degree $m \geq 3$ with energy slightly above the minimal energy $4\pi m$ are in fact global [6]. This should be contrasted with the wave map equation for which finite-time blow-up is possible in this class [9, 15], an indication that the blow-up question is more subtle for Schrödinger maps. By [2], degree $m = 1$ equivariant harmonic maps are unstable in the energy space, but stable in a stronger topology (which does not contradict [14]). A conditional global well-posedness result for Schrödinger maps appears in [17].

Various *local* existence results for the Cauchy problem (1.3) with k large enough are available [21, 13]. In light of the conservation laws (1.4) and (1.5), to prove Theorem 1, it will suffice to obtain an a priori estimate of $\|D^2 \vec{u}\|_{L^2}$ for smooth solutions – see Section 5.

A common strategy for estimating derivatives of maps, used in various geometric PDE contexts, is to express these derivatives – which lie tangent to the target space manifold – in a frame on the tangent space chosen so that the coordinates satisfy a “familiar” PDE, whose solutions can be estimated (*moving frames*). An example is the “generalized Hasimoto transform” of [4], in which the derivative of a *radial* solution $\vec{u}(r, t)$ of equation (1.1) is expressed in an orthonormal frame $\{\hat{e}(r, t), J^{\vec{u}(r, t)} \hat{e}(r, t)\}$ on $T_{\vec{u}(r, t)} \mathbb{S}^2$ which is parallel along $\vec{u}(r, t)$ for each t – that is $D_r e \equiv 0$ (D_r the covariant derivative):

$$T_{\vec{u}(r, t)} \mathbb{S}^2 \ni \vec{u}_r(r, t) = q_1(r, t) \hat{e}(r, t) + q_2(r, t) J^{\vec{u}(r, t)} \hat{e}(r, t),$$

and the resulting coordinates $q(r, t) = q_1(r, t) + iq_2(r, t)$ satisfy a cubic, non-local, Schrödinger equation:

$$iq_t = -\Delta q + \frac{1}{r^2} q + \left(\int_r^\infty |q(\rho, t)|^2 \frac{d\rho}{\rho} - \frac{1}{2} |q|^2 \right) q. \quad (1.6)$$

The precise version of this relation we use is:

Proposition 1. *There is a map $\vec{u} \mapsto q = q[\vec{u}]$ from radial maps with $\vec{u}(r) - \hat{k} \in H^2(\mathbb{R}^2)$ to complex radial functions $q(r)$ with $w(x) := e^{i\theta} q(r) \in H^1(\mathbb{R}^2)$ ((r, θ) polar coordinates on \mathbb{R}^2) such that if $\vec{u}(r, t)$ is a (radial) solution of (1.1), then $q(r, t) = q[\vec{u}]$ is a (radial) solution of (1.6). Further, the H^1 and H^2 norms of $\nabla \vec{u}$ and $w = e^{i\theta} q$ are comparable:*

$$\begin{cases} \|w(t)\|_{H^1(\mathbb{R}^2)} \lesssim \|\nabla \vec{u}(t)\|_{H^1(\mathbb{R}^2)} + \|\nabla \vec{u}(t)\|_{H^1(\mathbb{R}^2)}^2 \\ \|\nabla \vec{u}(t)\|_{H^1(\mathbb{R}^2)} \lesssim \|w(t)\|_{H^1(\mathbb{R}^2)} + \|w(t)\|_{H^1(\mathbb{R}^2)}^2. \end{cases} \quad (1.7)$$

$$\begin{cases} \|w(t)\|_{H^2(\mathbb{R}^2)} \lesssim \|\nabla \vec{u}(t)\|_{H^2(\mathbb{R}^2)} + \|\nabla \vec{u}(t)\|_{H^1(\mathbb{R}^2)}^3 \\ \|\nabla \vec{u}(t)\|_{H^2} \lesssim \|w(t)\|_{H^2(\mathbb{R}^2)} + \|w(t)\|_{H^1(\mathbb{R}^2)}^3. \end{cases} \quad (1.8)$$

Moreover, the map $\vec{u} \mapsto q$ is one-to-one: given two radial maps \vec{u}^A and \vec{u}^B as above, if the corresponding associated complex functions agree, $q^A \equiv q^B$, then so do the original maps, $\vec{u}^A \equiv \vec{u}^B$.

This is proved in Section 4.

Remark 1. It is natural to consider $w(x, t) = e^{i\theta} q(r, t)$ to handle the q/r^2 term in (1.6) (see eg. (3.4)). Notice regularity of w implies decay of q at $r = 0$.

Given this correspondence between equations (1.1) and (1.6), the main ingredient in proving Theorem 1 is an a priori estimate of $\|e^{i\theta} q(t)\|_{H^1}$ for solutions $q(r, t)$ of (1.6) with initial data $q_0(r) = q(r, 0)$ with $e^{i\theta} q_0(x) \in H^1$ ($q_0 \in H^1$, $q_0/r \in L^2$).

Remark 2. We take H^2 initial data in Theorem 1 so as to make the connection between equations (1.1) and (1.6) – as expressed in Proposition 1 – reasonably straightforward. Presumably, a more careful study of this connection could be used to lower the Sobolev index. We do not pursue it here.

So generalizing slightly, the heart of this paper is a study of the Cauchy problem for cubic, non-local, Schrödinger equations of this type, for radially-symmetric functions $q(r, t)$, in two space dimensions:

$$\begin{cases} i q_t = -\Delta q + \frac{1}{r^2} q + \left(K \int_r^\infty |q(\rho, t)|^2 \frac{d\rho}{\rho} - \frac{\lambda}{2} |q|^2 \right) q \\ q(r, 0) = q_0(r) \in L^2(\mathbb{R}^2), \end{cases} \quad (1.9)$$

where $\lambda \in \{0, \pm 1\}$, $K \in \mathbb{R}$. This family of equations includes:

- $\lambda = K = 0$: the free (linear) Schrödinger equation for functions of angular momentum one: $v(x, t) = e^{i\theta} q(r, t)$
- $K = 0$: the focusing ($\lambda = 1$) or defocusing ($\lambda = -1$) cubic Schrödinger equation (again, in the first angular momentum sector)
- $K = \lambda = 1$: equation (1.6) which, as discussed above, is satisfied by the derivative \vec{u}_r , as expressed in a particular frame, of a radial Schrödinger map from \mathbb{R}^2 to \mathbb{S}^2
- $K = -1, \lambda = -1$: an analagous equation for (radial) Schrödinger maps from \mathbb{R}^2 to hyperbolic space \mathbb{H}^2

Equation (1.9) formally preserves the L^2 -norm (or *mass*) of solutions,

$$\|q(t)\|_{L^2}^2 = \int_0^\infty |q(r, t)|^2 r \, dr = \|q_0\|_{L^2}^2,$$

and moreover is invariant under the L^2 -norm-preserving scaling $q(r, t) \mapsto Nq(Nr, N^2t)$ ($N > 0$), making this an L^2 -critical problem, and suggesting that global well-posedness may be a delicate issue. Indeed, for the (local) cubic NLS ($K = 0$), in the *focusing* case ($\lambda = 1$), it is well-known that solutions at or above a critical mass threshold may become singular in finite time, while solutions below this mass are global, as are *all* solutions in the *defocusing* ($\lambda = -1$) case: see [24] for H^1 data, [10] for L^2 data.

A crucial difference between (1.9) and its local ($K = 0$) counterpart, is that (1.9) has no conserved energy. A superficial consequence of this is a lack of obvious focusing/defocusing categorization for (1.9). However, a hint of its character can be

seen in the following formal (i.e. assuming solutions are smooth, with fast spatial decay at the origin and infinity) identities for solutions:

- *virial*-type identity:

$$\frac{d^2}{dt^2} \frac{1}{2} \int_0^\infty r^2 |q(r, t)|^2 r \, dr = \int_0^\infty \left\{ 4|q_r|^2 + 4 \frac{|q|^2}{r^2} + (2K - \lambda) |q|^4 \right\} r \, dr \quad (1.10)$$

- *Morawetz*-type identity:

$$\frac{d^2}{dt^2} \int_0^\infty r |q(r, t)|^2 r \, dr = \int_0^\infty \left\{ 3 \frac{|q|^2}{r^3} + \left(2K - \frac{\lambda}{2} \right) \frac{|q|^4}{r} \right\} r \, dr \quad (1.11)$$

which suggest (1.9) may have a defocusing character if, for example,

$$2K \geq \max \left(\lambda, \frac{\lambda}{2} \right). \quad (1.12)$$

For this reason, we might expect to have global well-posedness for (1.9), regardless of the size of the initial data, if (1.12) holds. This is our main result:

Theorem 2. *If (1.12) holds, then for any (radial) $q_0 \in L^2$, equation (1.9) has a unique global solution, which moreover scatters as $t \rightarrow \pm\infty$. If in addition $w_0(x) = e^{i\theta} q_0(r) \in H^k(\mathbb{R}^2)$ for $k = 1$ or 2 , then with $w(x, t) = e^{i\theta} q(r, t)$, $\|w(t)\|_{H^k(\mathbb{R}^2)}$ remains finite for all $t \geq 0$.*

This result may be of some wider interest, but in particular, the relation (1.12) indeed holds in the case $K = \lambda = 1$ of (1.6), and so Theorem 2 – in light of Proposition 1 – provides the estimates needed for the application to radial Schrödinger maps into \mathbb{S}^2 , and so proves Theorem 1.

Remark 3. For Schrödinger maps into hyperbolic space \mathbb{H}^2 , relation (1.12) does *not* hold, and global well-posedness is open.

In the absence of a conserved energy to control the H^1 norm (were we to take H^1 data), we approach the well-posedness of (1.9) in the celebrated framework for critical equations recently pioneered by Kenig-Merle [7], though naturally we follow most closely the work of Killip-Tao-Visan [10] on the 2D radial cubic (local) NLS.

Thus, we begin with the local theory:

- Proposition 2.** (1) *For each $q_0 \in L^2$, (1.9) has a unique solution $q \in C(I; L^2) \cap L^4_{loc}(I; L^4)$ on a maximal (and non-empty) time interval $I = [T_{min}, T_{max}] \ni 0$ (possibly $T_{min} = -\infty$ and/or $T_{max} = \infty$), which conserves the L^2 norm.*
- (2) *If $T_{max} < \infty$, then $\|q\|_{L^4_t([0, T_{max}]; L^4)} = \infty$ (an analogous statement holds for T_{min}).*
- (3) *If $T_{max} = \infty$ and $\|q\|_{L^4_t([0, \infty); L^4)} < \infty$, then q scatters as $t \rightarrow +\infty$ (an analogous statement holds for $t \rightarrow -\infty$).*
- (4) *The solution at each time depends continuously on the initial data. Further, the solution has the “stability” property as in Lemma 1.5 of [10].*
- (5) *If $\|q_0\|_{L^2}$ is sufficiently small, the solution is global ($I = (-\infty, \infty)$) and $\|q\|_{L^4_t(\mathbb{R}; L^4)} < \infty$.*

The proof is a mild variant of the proof in the local case [3, 22], as explained in Section 3.1.

In particular, there is global well-posedness for *small* L^2 data, and the approach of Kenig-Merle is to study a hypothetical solution that “blows up” in the sense that its space-time L^4 norm over its interval of existence is infinite,

$$\|q\|_{L^4(I;L^4)} = \infty \quad (1.13)$$

(a definition of “blowup” which includes merely non-scattering solutions, as well as those which fail to exist globally), and which does so with minimal L^2 norm (or *mass*). Such a solution is then shown to have strong compactness (and in [10], smoothness) properties. Our version is as follows:

Proposition 3. *If there is any L^2 data for which global well-posedness (or merely scattering) for (1.9) fails, then there is a solution $q(r, t)$, defined on maximal existence interval I , with minimal L^2 -mass among solutions blowing up as in (1.13), such that:*

- (1) *for $t \in I$, there is $N(t) \in (0, \infty)$ so that*

$$v(r, t) := N(t)^{-1}q(r/N(t), t)$$

is an L^2 pre-compact family (in t)

- (2) *we may assume q falls into one of the following three cases*

- *soliton-type solution: $I = \mathbb{R}$ and $N(t) \equiv 1$*
- *self-similar-type solution: $I = (0, \infty)$ and $N(t) = t^{-1/2}$*
- *inverse cascade-type solution: $I = \mathbb{R}$, $N(t) \lesssim 1$, $\liminf_{t \rightarrow -\infty} N(t) = \liminf_{t \rightarrow \infty} N(t) = 0$*

- (3) *$w(x, t) := e^{i\theta}q(r, t) \in H^s(\mathbb{R}^2)$ for every $s \geq 0$ and $t \in I$, and furthermore in the soliton and inverse cascade cases, $w \in L_t^\infty H^s(\mathbb{R}^2)$ for each $s \geq 0$.*

The proof of Proposition 3 follows the corresponding proof in [10] very closely. Because of the non-local nonlinearity, the estimates have to be done differently in a few key places – we give those details in Section 3.2.

After extracting a minimal blowup solution with nice properties, the Kenig-Merle strategy for proving global well-posedness (and scattering) is to rule out the possibility of the existence of such an object. This is the “equation specific” part of the program. In our case, the proof relies on modified versions of the identities (1.10) and (1.11), and so is in a sense similar to [7, 10]. However, there is a fundamental difference here – we have no conserved energy (which is typically what appears on the right-hand-side of an identity like (1.10)), and so we need finer estimates to get the contradiction. This is the main novelty of the paper, and is done in Section 2.

2. NON-EXISTENCE OF BLOWUP SOLUTIONS FOR THE NON-LOCAL NLS

In this Section we prove Theorem 2, modulo Proposition 3.2, by ruling out the possibility of soliton-, self-similar-, and inverse-cascade-type blowup solutions.

So let $q(r, t)$ be a minimal mass blowup solution on maximal existence interval I , with frequency scale function $N(t)$, furnished by Proposition 3, and set

$$w(x, t) = e^{i\theta}q(r, t).$$

And recall our standing assumption

$$2K \geq \max\left(\lambda, \frac{\lambda}{2}\right).$$

We will use a lower bound which follows easily from the compactness:

Lemma 2.1.

$$\|\nabla w(\cdot, t)\|_{L^2(\mathbb{R}^2)}^2 \sim \|q_r(\cdot, t)\|_{L^2}^2 + \|q(\cdot, t)/r\|_{L^2}^2 \gtrsim N^2(t).$$

Proof. First rescale $q(r, t) = N(t)v(N(t)r, t)$, and set $\tilde{w}(x, t) = e^{i\theta}v(r, t)$, so that the estimate we seek is $\|\nabla \tilde{w}(\cdot, t)\|_{L^2} \gtrsim 1$. If this fails, then for some sequence $\{t_n\}$, $\tilde{w}_n(x) := \tilde{w}(x, t_n)$, satisfies $\|\nabla \tilde{w}_n\|_{L^2(\mathbb{R}^2)} \rightarrow 0$. Since $\|\tilde{w}_n\|_{L^2} = \text{const.}$, we can extract a subsequence (still denoted \tilde{w}_n) with $\tilde{w}_n \rightarrow 0$ weakly in H^1 , and strongly in L^2 on disks. By the compactness, on the other hand, for any $0 < \eta$, $\|\tilde{w}_n\|_{L^2(\{|x| > C(\eta)\})} < \eta$, a contradiction. \square

2.1. The soliton case. Here $I = \mathbb{R}$ and $N(t) \equiv 1$.

The main tool is a spatially localized version of the virial identity (1.10). For a smooth cut-off function

$$\psi(r) \geq 0, \quad \psi \equiv 1 \text{ on } [0, 1), \quad \psi \equiv 0 \text{ on } [2, \infty),$$

and a fixed radius $R > 0$, define $\phi_R(r) := \psi(r/R)$, and the quantity

$$I_R(q) := \int_0^\infty r \operatorname{Im}(\bar{q} q_r) \phi_R r \, dr,$$

a function of time. By straightforward calculation we have

Lemma 2.2.

$$\begin{aligned} \frac{d}{dt} I_R(q) = 2 \int_0^\infty \left\{ |q_r|^2 + \frac{|q|^2}{r^2} + \mu |q|^4 \right. \\ + \left(|q_r|^2 + \frac{|q|^2}{r^2} + \mu |q|^4 \right) (\phi_R - 1) \\ + \left(|q_r|^2 - \frac{3}{4} \frac{|q|^2}{r^2} - \frac{\mu}{2} |q|^4 \right) r (\phi_R)_r \\ \left. - \frac{5}{4} \frac{|q|^2}{r^2} r^2 (\phi_R)_{rr} - \mu \frac{|q|^2}{r^2} r^3 (\phi_R)_{rrr} \right\} r \, dr. \end{aligned} \quad (2.1)$$

with $\mu := \frac{1}{4}(2K - \lambda) \geq 0$.

From Proposition 3 we have for each $s \geq 0$, and for all t ,

$$\|w(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^2)} \leq C_s. \quad (2.2)$$

Fix $\eta > 0$, and let $R = 2C(\eta)$ so that, since $N(t) \equiv 1$,

$$\int_{|x| > R/2} |w(x, t)|^2 dx < \eta \quad (2.3)$$

for all t . Multiplying w by a cut-off function $1 - \psi(2r/R)$, and interpolating between (2.3) and (2.2) with $s = 2$ (and using a Sobolev inequality) yields

$$\int_R^\infty \left\{ |q_r|^2 + \frac{|q|^2}{r^2} + \mu |q|^4 \right\} r dr \sim \int_{|x| \geq R} \{ |\nabla w|^2 + \mu |w|^4 \} dx \lesssim \eta^{1/2},$$

and so using $|1 - \phi_R|$, $|r(\phi_R)_r|$, $|r^2(\phi_R)_{rr}|$, $|r^3(\phi_R)_{rrr}| \lesssim 1$ in (2.1), we arrive at

$$\frac{d}{dt} I_R(q) \geq 2 \int_0^\infty \left\{ |q_r|^2 + \frac{|q|^2}{r^2} + \mu |q|^4 \right\} r \, dr - C\eta^{1/2}.$$

By Lemma 2.1 then, since $N(t) \equiv 1$, and for η chosen small enough,

$$\frac{d}{dt} I_R(q) \gtrsim 1.$$

On the other hand,

$$|I_R(q)| \lesssim R \|q\|_{L^2} \|q_r\|_{L^2} \lesssim RC_1.$$

These last two inequalities are in contradiction for sufficiently large t , and so the soliton-type blowup is ruled out.

2.2. The self-similar case. Here $I = (0, \infty)$, and $N(t) = t^{-1/2}$.

Again we use (2.1), but in this case, we need a stronger bound on the Sobolev norms – in fact, bounds which match Lemma 2.1. Such bounds follow from the regularity estimate of [10], in the self-similar case, as adapted to our non-local non-linearity in Section 3.3:

Lemma 2.3. *For any $s \geq 0$,*

$$\sup_{t \in (0, \infty)} \int_{|\xi| > At^{-1/2}} |\widehat{w}(\xi, t)|^2 d\xi \leq C_s A^{-s}, \quad A > A_0(s). \quad (2.4)$$

As a consequence,

$$\|w(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^2)} \lesssim t^{-s/2} = [N(t)]^s. \quad (2.5)$$

Indeed, after re-scaling $w(x, t) = N(t)\tilde{w}(N(t)x, t)$, equation (2.4) reads

$$\int_{|\xi| > A} |\widehat{\tilde{w}}(\xi, t)|^2 d\xi \leq C_s A^{-s}$$

for all t , from which follows $\|\tilde{w}\|_{\dot{H}^s} \lesssim 1$, and thus (undoing the scaling) (2.5).

Now fix a small $\eta > 0$, and large T . A (localized) interpolation (just as in Section 2.1) between (2.5) with $s = 2$ and the L^2 smallness from compactness, gives

$$\int_{2C(\eta)/N(t)}^\infty \left\{ |q_r|^2 + \frac{|q|^2}{r^2} + \mu |q|^4 \right\} r \, dr \lesssim \eta^{1/2} \|w\|_{\dot{H}^2(\mathbb{R}^2)} \lesssim \eta^{1/2} (N(t))^2.$$

Using this, with η small enough, and Lemma 2.1, in (2.1), we find, for $t < T$, and $R = 2C(\eta)/N(T) > 2C(\eta)/N(t)$,

$$\frac{d}{dt} I_R(q) \gtrsim N^2(t) = \frac{1}{t},$$

and hence for $T \gg 1$,

$$I_R(q)(T) \gtrsim I_R(q)(1) + \int_1^T \frac{dt}{t} \gtrsim \log(T).$$

On the other hand

$$|I_R(q)(T)| \lesssim R \|q(T)\|_{L^2} \|q(T)\|_{\dot{H}^1} \lesssim \frac{C(\eta)}{N(T)} N(T) = C(\eta).$$

The last two inequalities are in contradiction for T large enough, and so the self-similar-type blowup is ruled out.

2.3. The inverse-cascade case. Here $I = \mathbb{R}$, $N(t) \lesssim 1$, and $\liminf_{t \rightarrow -\infty} N(t) = \liminf_{t \rightarrow \infty} N(t) = 0$.

The main tool is a variant of the Morawetz identity (1.11). Set

$$\psi(r) := \begin{cases} 4r - r^2 & 0 < r \leq 1 \\ 6 - \frac{4}{r} + \frac{1}{r^2} & 1 < r < \infty \end{cases}.$$

It is easily checked that for $r \in (0, \infty)$,

- $\psi \in C^3$
- $0 < \psi < 6$
- $\psi_r > 0$
- $\alpha(r) := \frac{1}{2}\psi_r + \frac{3}{2}\frac{\psi}{r} - r\psi_{rr} - \frac{1}{2}r^2\psi_{rrr} > 0$
- $\beta(r) := \frac{\psi}{r} - \psi_r > 0$.

Set

$$P(q) := \int_0^\infty \operatorname{Im}(\overline{q}q_r)\psi(r)r \, dr.$$

For solutions of (1.9), an elementary computation gives:

Lemma 2.4.

$$\frac{d}{dt}P(q) = \int_0^\infty \left\{ 2\psi_r|q_r|^2 + \alpha(r)\frac{|q|^2}{r^2} + \left(\frac{K}{2}\beta(r) + \mu\left(\frac{\psi}{r} + \psi_r\right) \right) |q|^4 \right\} r \, dr > 0. \quad (2.6)$$

Note that since $|\psi(r)| \lesssim 1$,

$$|P(q)| \lesssim \|q\|_{L^2}\|q_r\|_{L^2} \lesssim \|q_r\|_{L^2}. \quad (2.7)$$

Next recall that for some sequences $t_n \rightarrow -\infty$, $T_n \rightarrow +\infty$, $N(t_n) \rightarrow 0$ and $N(T_n) \rightarrow 0$. It then follows easily from the definition of $N(t)$ that $\|q_r(t_n)\|_{L^2} \rightarrow 0$ and $\|q_r(T_n)\|_{L^2} \rightarrow 0$. Hence by (2.7),

$$P(q(t_n)) \rightarrow 0, \quad P(q(T_n)) \rightarrow 0. \quad (2.8)$$

If $P(q_0) \geq 0$, then (2.6) implies $P(q(t)) > 0$ and increasing for $t > 0$, while if $P(q_0) < 0$, then (2.6) implies $P(q(t)) < 0$ and increasing for $t < 0$. In either case, (2.8) is contradicted. This rules out the inverse cascade-type blowup.

Proof of Theorem 2: Having ruled out the possible blow-up scenarios, Proposition 2 gives global well-posedness (and scattering) for $q_0 \in L^2$. In particular, using $\|q\|_{L^4_{x,t}(\mathbb{R}^2 \times [0, \infty))} < \infty$, and applying Strichartz estimates to derivatives of (1.9), one obtains, in a standard way, the “propagation of regularity”: if $e^{i\theta}q_0(r) \in H^k$ for $k = 1$ or $k = 2$, then $e^{i\theta}q(r, t)$ is bounded in H^k on bounded time intervals (see, eg., [4]). \square

3. MINIMAL MASS BLOWUP SCENARIOS

Here we present the proofs of Proposition 2 and Proposition 3, following [10] very closely. Indeed, nearly the entire argument in Sections 4-7 of [10] for the *local* radial cubic NLS carries over, line-by-line, to our *non-local* equation (1.9). So we give here only a rough outline of the arguments, emphasizing details only where they differ significantly from [10].

Remark 4. Of course the method of *ruling out* blow-up used in [10] relies on conservation of energy, and so does *not* apply to (1.9) – hence the alternative method given in Section 2.

In most places where the nonlinearity needs to be estimated, the elementary Hardy-type inequality for radial functions

$$\|f(r)\|_{L^p} \lesssim \|rf_r\|_{L^p}, \quad 1 \leq p < \infty \quad (3.1)$$

(used for this purpose in [4], for example), together with Hölder, suffices to handle the non-local term: for $1 \geq \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$, $\frac{1}{s} = \frac{1}{p_2} + \frac{1}{p_3} > 0$,

$$\begin{aligned} \left\| q_1(r) \int_r^\infty |q_2(\rho)| |q_3(\rho)| \frac{d\rho}{\rho} \right\|_{L^p} &\leq \|q_1\|_{L^{p_1}} \left\| \int_r^\infty |q_2(\rho)| |q_3(\rho)| \frac{d\rho}{\rho} \right\|_{L^s} \\ &\lesssim \|q_1\|_{L^{p_1}} \|q_2\|_{L^{p_2}} \|q_3\|_{L^{p_3}}. \end{aligned} \quad (3.2)$$

Another convenience is to work with the function $w(x, t)$ given in polar coordinates by

$$w(x, t) = e^{i\theta} q(r, t), \quad (3.3)$$

for which equation (1.9) becomes

$$iw_t = -\Delta w + \left(K \int_{|y| \geq |x|} \frac{|w(y)|^2}{|y|^2} dy - \frac{\lambda}{2} |w|^2 \right) w. \quad (3.4)$$

The advantage is that here the true Laplacian replaces $\Delta - 1/r^2$ in (1.9). While $w(x, t)$ is not radial, all the estimates in [10] which require radial symmetry, apply also to functions of the form (3.3) (equivalently, replacing Δ by $\Delta - 1/r^2$ on radial functions – which indeed generally only *improves* decay at the origin), as we shall explain below. Throughout this section we will use both representations $q(r, t)$ and $w(x, t) = e^{i\theta} q(r, t)$, and corresponding equations (1.9) and (3.4), as needed.

3.1. The local theory. *Proof of Proposition 2.* Except for the “stability” statement, the proof is a mild extension of the proof of the classical result of Cazenave-Weissler (see [3]) for the (local) cubic NLS, as applied to equation (3.4) for $w(x, t)$. The main ingredient is the Strichartz estimate, our version of which follows from a version of (3.2) (with $p = 4/3$) generalized to $w(x, t)$:

$$\begin{aligned} \left\| w \int_{|y| \geq |x|} \frac{|w(y)|^2}{|y|^2} dy \right\|_{L_x^{4/3}} &\lesssim \|w\|_{L_x^4} \left\| \int_{|y| \geq r} \frac{|w(y)|^2}{|y|^2} dy \right\|_{L_x^2} \\ &\lesssim \|w\|_{L_x^4} \left\| r \frac{\partial}{\partial r} \int_0^\infty \frac{dr}{r} \int_0^{2\pi} |w(r, \theta)|^2 d\theta \right\|_{L_x^2} \\ &= \|w\|_{L_x^4} \left\| \int_0^{2\pi} |w(r, \theta)|^2 d\theta \right\|_{L_x^2} \lesssim \|w\|_{L_x^4}^3 \end{aligned}$$

(using Hölder in the last step), which leads to an estimate of the non-local nonlinear term in dual Strichartz spaces, which is the same as that for the local cubic term:

$$\left\| w_1 \int_{|y| \geq r} \frac{|w_1(y)|^2}{|y|^2} dy - w_2 \int_{|y| \geq r} \frac{|w_2(y)|^2}{|y|^2} dy \right\|_{L_{x,t}^{4/3}} \lesssim \left[\|w_1\|_{L_{x,t}^4}^2 + \|w_2\|_{L_{x,t}^4}^2 \right] \|w_1 - w_2\|_{L_{x,t}^4}. \quad (3.5)$$

We omit the details. Similarly, this inequality can be used to prove the “stability” statement as a straightforward modification of the proof in [22, Lemma 3.6] for the local case. \square

3.2. The blow-up scenarios. The proof in [10, Section 4] of the existence of a minimal-mass blowup solution, pre-compact modulo scaling, of soliton, self-similar, or inverse-cascade type – that is, of the the first two statements of Proposition 3 – applies nearly without modification to (1.9).

The proof of existence of a pre-compact minimal blow-up solution in [23] applies here. It rests primarily on the local theory, Proposition (2), the cubic scaling, and linear estimates (radial symmetry plays no role). Where nonlinear estimates come in, in particular the “asymptotic solvability” [23, Lemma 5.2], they are easily handled using (3.2). Thus we have Proposition 3, part (1).

The argument of [10, Section 4], which again rests on the local theory and the scaling, and does not use radial symmetry, then carries over directly to give us part (2) of Proposition 3.

The [10] proof of the regularity statements – part (3) of Proposition 3 – depends much more heavily on nonlinear estimates and radial symmetry. We explain its adaptation to our setting in the next two sub-sections.

3.3. Regularity of self-similar blowups. Here we work with $w(x, t) = e^{i\theta} q(r, t)$, a minimal-mass blowup solution of self-similar type. So $t \in (0, \infty)$, and $N(t) = t^{-1/2}$.

The arguments in [10, Section 5] are used to show that for all $t > 0$, $s > 0$, and A large enough,

$$\|w_{>At^{-1/2}}(t)\|_{L^2} \lesssim_{s,w} A^{-s}, \quad (3.6)$$

where $w_{>N}$ denotes the Littlewood-Paley projection of w onto frequencies above N . In particular, this gives $w(t) \in H^s$ for all $s \geq 0$, and all $t > 0$ – that is, Part (3) of Proposition 3. It is rephrased as the Lemma 2.3 we used in Section 2.2.

The adaptation to our setting of the nonlinear estimate [10, Lemma 5.3] requires some comment. Key to this argument is a decomposition of w into high-, medium-, and low-frequency components:

$$w = w_{>(A/8)T^{-1/2}} + w_{\sqrt{AT}^{-1/2} < \cdot \leq (A/8)T^{-1/2}} + w_{\leq \sqrt{AT}^{-1/2}}. \quad (3.7)$$

First, note that this decomposition preserves the form of function $w(x) = e^{i\theta} q(r)$ (each term is $e^{i\theta}$ multiplying a radial function).

Second, note that the non-local nonlinearity behaves well with respect to frequency decomposition. Denoting

$$I(f)(r) := \int_r^\infty f(\rho) \frac{d\rho}{\rho}$$

for a radial function $f(r)$, we have $x \cdot \nabla I = r I_r = -f$, so

$$\widehat{f} = -\nabla_\xi \cdot \xi \widehat{I} = -|\xi|^{-1} \partial_{|\xi|} |\xi|^2 \widehat{I}$$

and

$$\widehat{I}(|\xi|) = \frac{1}{|\xi|^2} \int_{|\xi|}^\infty \widehat{f}(|\eta|) |\eta| d|\eta|.$$

Hence if f is frequency localized in a particular disk, so is $I(f)$. So after decomposing w as in (3.7), one can assume, exactly as in [10, Lemma 5.3], that each term of the resulting expansion of the high frequency projection of the nonlinearity, $P_{>AT-1/2}(wI(|w|^2))$, must somewhere include the high frequency component $w_{>(A/8)T-1/2}$.

The estimates in this Lemma then carry over, using (3.2) as needed, with one exception: the use of the bilinear Strichartz inequality to estimate nonlinear terms containing two low-frequency factors. The problem occurs in the non-local nonlinear term when the high-frequency factor fall outside the integral, as in

$$w_{>(A/8)T-1/2}I(|w_{\leq\sqrt{A}T-1/2}|^2).$$

This term does not involve a (local) product of a low-frequency and a high-frequency “approximate solution” of the Schrödinger equation, and so it is unclear how to apply the bilinear Strichartz estimate to it.

We can get around this problem by replacing the use of bilinear Strichartz with an application Shao’s Strichartz estimate for radial functions [16]

$$\|P_N e^{it\Delta} f\|_{L_{x,t}^q(\mathbb{R}\times\mathbb{R}^2)} \lesssim N^{1-4/q} \|f\|_{L^2(\mathbb{R}^2)}, \quad q > 10/3 \quad (3.8)$$

plus a Bernstein estimate.

Remark 5. Note that (3.8) is for radial functions, while our functions are of the form $w(x) = e^{i\theta}q(r)$. In fact it is easily checked that Shao’s argument applies also for such functions – it is essentially a matter of replacing the Bessel function J_0 with J_1 , which has the same spatial asymptotics (and better behaviour at the origin). The same is true for the weighted Strichartz estimate [10, Lemma 2.7], which is also used in the [10] argument we are following.

Indeed, since $I(|w_{\leq M}|^2)$ is frequency-localized below M , applying Hölder, Shao, Bernstein, and Hardy, we have, for any $10/3 < q < 4$

$$\begin{aligned} \|IP_N e^{it\Delta} f\|_{L_{x,t}^{4/3}} &\lesssim \|I\|_{L_{x,t}^{\frac{4q}{3q-4}}} \|P_N e^{it\Delta} f\|_{L_{x,t}^q} \lesssim M^{\frac{4}{q}-1} \|I\|_{L_t^{\frac{4q}{3q-4}} L_x^{\frac{4q}{q+4}}} N^{1-\frac{4}{q}} \|P_N f\|_{L^2} \\ &= \left(\frac{M}{N}\right)^{\frac{4}{q}-1} \|w_{\leq M}\|_{L_t^{\frac{8q}{3q-4}} L_x^{\frac{8q}{q+4}}}^2 \|P_N f\|_{L^2}, \end{aligned}$$

and the middle factor is a Strichartz norm, so is bounded by a constant. By this argument, using also the inhomogeneous version of (3.8) (which follows in the usual way), and replacing P_N by $P_{\geq N}$ (which follows easily by summing over dyadic frequencies), we can finally arrive at the nonlinear estimate [10, Lemma 5.3], albeit with a slower decay factor $A^{-(2/q-1/2)}$ replacing $A^{-1/4}$ (notice $0 < 2/q-1/2 < 1/10$). This lower power does not matter, however, and the remaining estimates from Section 5 of [10] carry through, to establish the desired estimate (3.6).

3.4. Regularity of soliton and inverse-cascade blowups. The [10, Section 6] proof of regularity for the global cases – the soliton- and inverse-cascade-type blowup solutions – relies heavily on the radial symmetry, particularly through a decomposition of solutions into “incoming” and “outgoing” waves, defined by projections with Bessel function kernels. These projections are defined analogously for functions $w(x) = e^{i\theta}q(r)$ by simply replacing the Bessel (and Hankel) functions of order zero

with those of order one: $J_0 \rightarrow J_1$, $H_0^\alpha \rightarrow H_1^\alpha$. It is easily checked that these (new) projections obey the kernel estimates listed in [10, Proposition 6.2], essentially because J_1 and H_1 have the same behaviour as J_0 and H_0 away from the origin [10, eqns. (77), (79)]. (At the origin, J_1 is *better* behaved, while H_1 is worse – though this plays no role in the estimates.)

Given this, the subsequent estimates of [10, Section 7] all carry over to our case, as above using (3.2) where needed to estimate the non-local nonlinearity, to establish $w \in L_t^\infty H_x^s$ for any $s > 0$.

This completes the proof of Proposition 3. \square

4. RELATING SCHRÖDINGER MAPS TO THE NONLOCAL NLS

Here we prove Proposition 1.

4.1. Construction of the frame. Following [4], given a radial map $\vec{u}(r) \in \widehat{k} + H^k$, we want to construct a unit tangent vector field, parallel transported along the curve $\vec{u}(r) \in \mathbb{S}^2$:

$$\widehat{e}(r) \in T_{\vec{u}(r)}\mathbb{S}^2, \quad |\widehat{e}| \equiv 1, \quad D_r \widehat{e}(r) \equiv 0, \quad (4.1)$$

where here D denotes covariant differentiation of tangent vector fields: given $\vec{\xi}(s) \in T_{\vec{u}(s)}\mathbb{S}^2$,

$$D_s \vec{\xi}(s) = P_{T_{\vec{u}(s)}\mathbb{S}^2} \partial_s \vec{\xi}(s) = \partial_s \vec{\xi}(s) + (\partial_s \vec{u}(s) \cdot \vec{\xi}(s)) \vec{u}(s) \in T_{\vec{u}(s)}\mathbb{S}^2.$$

Since we have fixed the boundary condition (at infinity) $\vec{u}(r) \rightarrow \widehat{k}$ as $r \rightarrow \infty$ (at least in the L^2 sense), we fix a unit vector in $T_{\widehat{k}}\mathbb{S}^2$, say $\widehat{i} = (1, 0, 0)$ to be the boundary condition for \widehat{e} (at infinity) and write

$$\widehat{e}(r) = \widehat{i} + \tilde{e}(r), \quad \vec{u}(r) = \widehat{k} + \tilde{u}(r)$$

so that the parallel transport equation $D_r \widehat{e} \equiv 0$ becomes

$$\tilde{e}_r = -(\tilde{u}_r \cdot [\widehat{i} + \tilde{e}(r)])(\widehat{k} + \tilde{u}) = -(\tilde{u}_1)_r \widehat{k} - (\tilde{u} \cdot \tilde{e}) \vec{u} - (\tilde{u}_r \cdot \widehat{i}) \tilde{u}, \quad (4.2)$$

which we will therefore solve in from infinity as

$$\tilde{e}(r) = -\tilde{u}_1(r) \widehat{k} + \int_r^\infty \left\{ (\tilde{u}(s) \cdot \tilde{e}(s)) \vec{u}(s) - (\tilde{u}_r(s) \cdot \widehat{i}) \tilde{u}(s) \right\} ds =: M(\tilde{e})(r) \quad (4.3)$$

by finding a fixed point of the map M in the space $X_R^2 := L_{rdr}^2([R, \infty); \mathbb{R}^3)$ for R large enough. To this end, we need the simple estimate

Lemma 4.1.

$$\left\| \int_r^\infty f(s) ds \right\|_{X_R^2} \leq \|f\|_{L_{rdr}^1([R, \infty))} =: \|f\|_{X_R^1}.$$

Proof. First by Hölder, for $r \geq R$,

$$\left| \int_r^\infty f(s) ds \right| = \left| \int_r^\infty \frac{1}{s} f(s) s ds \right| \leq \frac{1}{r} \|f\|_{X_R^1}. \quad (4.4)$$

Next, setting $F(r) := \int_r^\infty f(s)ds$ so $F' = -f$, we have $F^2(r) = 2 \int_r^\infty F(s)f(s)ds$, so changing order of integration and using (4.4),

$$\begin{aligned} \|F\|_{X_R^2}^2 &= 2 \int_R^\infty r dr \int_r^\infty F(s)f(s)ds \leq 2 \int_R^\infty |F(s)| |f(s)| ds \int_R^s r dr \\ &\leq \int_R^\infty |F(s)| s |f(s)| s ds \leq \sup_{r \geq R} (r|F(r)|) \|f\|_{X_R^2} \leq \|f\|_{X_R^1} \|f\|_{X_R^2} \end{aligned}$$

and the proof is completed by dividing through by $\|f\|_{X_R^2}$. \square

Now we may use Lemma 4.1 to estimate the map M :

$$\|M(\tilde{e})\|_{X_R^2} \leq \|\tilde{u}\|_{X_R^2} + \|\tilde{u}(s)(|\tilde{e}(s)| + |\tilde{u}_r(s)|)\|_{X_R^1} \leq \|\tilde{u}\|_{X_R^2} + \|\tilde{u}\|_{X_R^2} \|\tilde{e}\|_{X_R^2} + \|\tilde{u}\|_{X_R^2} \|\tilde{u}_r\|_{X_R^2}.$$

Since $\tilde{u} \in H^1(\mathbb{R}^2)$, there is R_0 such that for $R \geq R_0$, $\|\tilde{u}\|_{X_R^2} < 1/3$ and $\|\tilde{u}\|_{X_R^2} \|\tilde{u}_r\|_{X_R^2} < 1/3$, so

$$\|\tilde{e}\|_{X_R^2} \leq 1 \implies \|M(\tilde{e})\|_{X_R^2} \leq 1,$$

that is, M sends the unit ball in X_R^2 to itself. Also, for any $\tilde{e}^A, \tilde{e}^B \in X_R^2$,

$$\|M(\tilde{e}^A) - M(\tilde{e}^B)\|_{X_R^2} \leq \|\tilde{u}\|_{X_R^2} \|\tilde{e}^A - \tilde{e}^B\|_{X_R^2} < \frac{1}{3} \|\tilde{e}^A - \tilde{e}^B\|_{X_R^2},$$

so M is a contraction on the unit ball in X_R^2 , hence has a unique fixed point there.

Using $\tilde{u} \in H^2(\mathbb{R}^2)$, it follows from (4.2), that $\tilde{e}_r \in X_R^2$, $\tilde{e}/r \in X_R^2$, and, after differentiating once, $\tilde{e}_{rr} \in X_R^2$. In particular, \tilde{e} is continuously differentiable, so a genuine solution of (4.2).

Now we may simply solve the initial value problem for the linear ODE (4.2) from $r = R$ (with value $\tilde{e}(R)$) down to $r = 0$ to get \tilde{e} on $(0, \infty)$. Estimates as above imply that $\tilde{e} \in H^2(\mathbb{R}^2)$ (and in particular is continuous, and defined at $r = 0$). It is easily shown that if, in addition, $\tilde{u} \in H^3(\mathbb{R}^2)$, then $\tilde{e} \in H^3(\mathbb{R}^2)$.

So we have constructed a solution $\widehat{e}(r) = \widehat{i} + \tilde{e}(r)$ of $D_r \widehat{e} \equiv 0$. It then follows directly from this ODE that $\partial_r(\tilde{u}(r) \cdot \widehat{e}(r)) \equiv 0$ and $\partial_r(\widehat{e} \cdot \widehat{e}) \equiv 0$ and hence that $\widehat{e}(r) \in T_{\tilde{u}(r)} \mathbb{S}^2$ and $|\widehat{e}(r)| \equiv 1$. So we have (4.1).

4.2. Equation for $q(r, t)$. Given a radial Schrödinger map $\vec{u}(r, t)$ on a time interval $t \in [0, T]$ with $\vec{u}(\cdot, t) \in \widehat{k} + H^k(\mathbb{R}^2)$ for $k = 2$ or $k = 3$, for each $t \in [0, T]$ we construct the vector field \widehat{e} as above, yielding $\widehat{e}(r, t)$. By the ODE (4.3) for \tilde{e} , $\widehat{e}(r, t)$ has the same time-regularity as $\vec{u}(r, t)$ – i.e. $\widehat{e}_t \in L_t^\infty H^{k-2}(\mathbb{R}^2)$.

Now for each r and t , $\widehat{e}(r, t)$ and $J^{\vec{u}(r, t)} \widehat{e}(r, t) = \vec{u}(r, t) \times \widehat{e}(r, t)$ form an orthonormal frame on $T_{\vec{u}(r, t)} \mathbb{S}^2$, and so we may express

$$T_{\vec{u}(r, t)} \mathbb{S}^2 \ni \vec{u}_r(r, t) = q_1(r, t) \widehat{e}(r, t) + q_2(r, t) J^{\vec{u}(r, t)} \widehat{e}(r, t).$$

It is shown in [4] that the complex-valued function $q(r, t) = q_1(r, t) + iq_2(r, t)$ then satisfies equation (1.6) (we will not repeat the derivation here).

4.3. Equivalence of norms. We have

$$\vec{u}_r = q_1 \widehat{e} + q_2 J \widehat{e} =: q \circ \widehat{e}$$

(the last equality just defines a convenient notation), so

$$|q| = |\vec{u}_r|,$$

and since $D_r \widehat{e} \equiv 0$,

$$q_r \circ \widehat{e} = D_r(q \circ \widehat{e}) = D_r \vec{u}_r = \vec{u}_{rr} + |\vec{u}_r|^2 \vec{u}$$

so

$$|q_r| \leq |\vec{u}_{rr}| + |\vec{u}_r|^2, \quad |\vec{u}_{rr}| \leq |q_r| + |q|^2.$$

Setting $w(x) = e^{i\theta} q(r)$, and taking norms:

$$\|w\|_{H^1(\mathbb{R}^2)} \lesssim \|q_r\|_{L^2} + \|q/r\|_{L^2} \lesssim \|\vec{u}_{rr}\|_{L^2} + \|\vec{u}_r\|_{L^4}^2 + \|\vec{u}_r/r\|_{L^2} \lesssim \|\nabla \vec{u}\|_{H^1(\mathbb{R}^2)} + \|\nabla \vec{u}\|_{H^1(\mathbb{R}^2)}^2$$

(using a Sobolev inequality at the end). And in the opposite direction,

$$\|\nabla \vec{u}\|_{H^1(\mathbb{R}^2)} \lesssim \|\vec{u}_{rr}\|_{L^2} + \|\vec{u}_r/r\|_{L^2} \lesssim \|q_r\|_{L^2} + \|q\|_{L^4}^2 + \|q/r\|_{L^2} \lesssim \|w\|_{H^1(\mathbb{R}^2)} + \|w\|_{H^1(\mathbb{R}^2)}^2.$$

These last two inequalities give (1.7). Taking another covariant derivative in r and proceeding in a similar way yields (1.8).

4.4. One-to-one. Suppose $\vec{u}^A(r)$ and $\vec{u}^B(r)$ are two maps in $\widehat{k} + H^2(\mathbb{R}^2)$, and let $\widehat{e}^A(r)$, $\widehat{e}^B(r)$, and $q^A(r)$, $q^B(r)$ be the corresponding unit tangent vector fields, and complex functions (respectively) constructed as above. If we also denote $\widehat{f} := J\widehat{e}$, we have the linear ODE system

$$\frac{d}{dr} \begin{pmatrix} \vec{u} \\ \widehat{e} \\ \widehat{f} \end{pmatrix} = \begin{pmatrix} 0 & q_1 & q_2 \\ -q_1 & 0 & 0 \\ -q_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ \widehat{e} \\ \widehat{f} \end{pmatrix} =: A(q) \begin{pmatrix} \vec{u} \\ \widehat{e} \\ \widehat{f} \end{pmatrix}.$$

Suppose now that $q^A(r) \equiv q^B(r) =: q(r)$. Then we have

$$W := \begin{pmatrix} \vec{u}^A \\ \widehat{e}^A \\ \widehat{f}^A \end{pmatrix} - \begin{pmatrix} \vec{u}^B \\ \widehat{e}^B \\ \widehat{f}^B \end{pmatrix} \in H^2(\mathbb{R}^2), \quad W_r = A(q)W.$$

Applying the estimate of Lemma 4.1, we find

$$\|W\|_{L^2_{rdr}[R,\infty)} \leq C \|W_r\|_{L^1_{rdr}[R,\infty)} \leq C \|q\|_{L^2_{rdr}[R,\infty)} \|W\|_{L^2_{rdr}[R,\infty)}.$$

Choosing R large enough so that $\|q\|_{L^2_{rdr}[R,\infty)} < 1/C$, we conclude $W \equiv 0$ on $[R, \infty)$. Then standard uniqueness for initial value problems for linear ODE implies $W(r) \equiv 0$ for all r .

This completes the proof of Proposition 1. \square

5. GLOBAL SCHRÖDINGER MAPS

Theorem 2, together with Proposition 1, provides the a priori bounds on solutions of (1.3) needed to prove Theorem 1, via a standard approximation argument, as we explain here.

Proof of Theorem 1: Problem (1.3) is known to be locally well-posed for smoother initial data. In particular, for $\nabla \vec{u}_0 \in H^2$, [13] furnishes a unique local solution of (1.3) with $\nabla \vec{u} \in L^\infty([0, T]; H^2(\mathbb{R}^2))$, which may be continued as long as $\|\nabla \vec{u}(t)\|_{H^2(\mathbb{R}^2)}$ remains finite. This solution conserves energy (1.4) and furthermore, by (1.5), if $\vec{u}_0 - \widehat{k} \in L^2$, then $\|\vec{u}(t) - \widehat{k}\|_{L^2}$ remains constant.

Let $w(x, t) = e^{i\theta} q(r, t) \in L^\infty([0, T]; H^2(\mathbb{R}^2))$ be the corresponding solution of (1.6) furnished by Proposition 1. By Theorem 2, the solution can be extended globally, and moreover satisfies a bound of the form

$$\|w(\cdot, t)\|_{H^1(\mathbb{R}^2)} \leq C_{\|w(\cdot, 0)\|_{H^1}}(t) < \infty.$$

So invoking Proposition 1 again, we find

$$\|\vec{u}(\cdot, t) - \widehat{k}\|_{H^2(\mathbb{R}^2)} \leq C_{\|\vec{u}_0 - \widehat{k}\|_{H^2}}(t) < \infty. \quad (5.1)$$

Now suppose merely $\vec{u}_0 - \widehat{k} \in H^2(\mathbb{R}^2)$. Approximate $\vec{u}_0 - \widehat{k}$ in $H^2(\mathbb{R}^2)$ by maps $\vec{u}_0^j - \widehat{k} \in H^3(\mathbb{R}^2)$ (this can be done maintaining the constraint $|\vec{u}_0^j| \equiv 1$ since $H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$), and let $\vec{u}^j(t)$ be the corresponding global solutions of (1.3). Using the a priori bound (5.1) for \vec{u}^j , a standard argument (see, eg. [3]) shows that one can pass to a limit and obtain a solution $\vec{u}(x, t)$ of (1.3) with $\vec{u} - \widehat{k} \in L_{loc}^\infty([0, \infty); H^2(\mathbb{R}^2))$.

Finally, uniqueness of this solution follows from Proposition 1 and uniqueness of solutions of the cubic NLS (1.9). \square

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